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I Semester M.Sc. Examination, February - 2020
(CBCS-Y2K17/Y2K14 Scheme)

MATHEMATICS

M102T : Real Analysis

Time : 3 Hours

Max. Marks : 70

Instructions : *(i)* *Answer **any five** questions.*
(ii) ***All** questions carry **equal** marks.*

1. (a) Show that $f(x) = -x \in R[-c, 0]$. **4+5+5**

(b) If $f \in R[\alpha]$ on [a, b] and P, P* are two partitions of [a, b] such that $P \subset P^*$, then P.T. $L(P, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P, f, \alpha)$.

(c) If $f \in R[\alpha_1]$ on [a, b] and $f \in R[\alpha_2]$ on [a, b] then prove that $f \in R[\alpha_1 + \alpha_2]$ on [a, b].

2. (a) If $f \in R[\alpha]$ on [a, b] and $c \in R^+$, then prove that $cf \in R[\alpha]$ on [a, b]. **4+4+6**

(b) Assuming $f(x)$ is monotonic on [a, b] and $\alpha(x)$ is monotonically increasing and continuous function on [a, b], prove that $f \in R[\alpha]$ on [a, b].

(c) Let f be Riemann integrable on [a, b] and let $F(x) = \int_a^x f(t)\, dt$, where $a \leq x \leq b$. Then prove that F is continuous on [a, b]. Further, show that if $f(t)$ is continuous at a point x_0 on [a, b], then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

3. (a) If $f \in R[a, b]$ and if there exists a function F on [a, b] such that $F' = f$, then prove that $\int_a^b f\, dx = F(b) - F(a)$. **5+4+5**

(b) If $\underset{\mu(P) \to 0}{\text{lt}} S(P, f, \alpha)$ exists then prove that $f \in R[\alpha]$ on [a, b] and that

$$\underset{\mu(P) \to 0}{\text{lt}} S(P, f, \alpha) = \int_a^b f\, d\alpha$$

(c) Define a function of bounded variation. Prove that a function of bounded variation on [a, b] is bounded.

P.T.O.

4. (a) State and prove Weierstrauss M-test. **5+4+5**

(b) Test for uniform convergence for $\left\{\frac{nx}{1+n^2x^2}\right\}$ on [0, 1].

(c) Suppose $f_n \to f$ uniformly on [a, b] and if $x_0 \in [a, b]$ such that $\lim_{x \to x_0} f_n(x) = A_n$, n = 1, 2, 3, ... then prove that

(i) A_n converges

(ii) $\lim_{x \to x_0} \operatorname{lt}_{n \to \infty} f_n(x) = \operatorname{lt}_{n \to \infty} \lim_{x \to x_0} f_n(x)$

5. (a) If $|f_n(x)| < M_n$, $\forall n \in N$, $\forall x \in [a, b]$ and $\sum_{n=1}^{\infty} M_n$ of positive reals, is convergent, **5+4+5**

then prove that $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on [a, b].

(b) Show that $\sum_{n=1}^{\infty} nx\, e^{-nx^2}$ converges point-wise and not uniformly on [0, 4], k > 0.

(c) Let $\sum_{n=0}^{\infty} f_n(x)$ be an infinite series of functions uniformly convergent to $f(x)$ on [a, b] and each $f_n(x) \in R[a, b]$ then prove that $f(x) \in R[a, b]$.

Also, prove that $\int_a^x \left\{\sum_{n=1}^{\infty} f_n(t)\right\} dt = \sum_{n=k}^{\infty} \left\{\int_a^x f_n(t)dt\right\}$.

6. (a) If A is a sub-set of R. Then prove that the following statements are equivalent. **8+6**

(i) A is closed and bounded

(ii) A is compact

(iii) A is countably compact

(b) Prove that any infinite bounded subset of $\mathbf{R}^k$ has a limit point in $\mathbf{R}^k$.

7. (a) Let $E \subset \mathbf{R}^n$ be an open set and $f: E \to \mathbf{R}^m$ be a map. Prove that f is continuously differentiable if and only if the partial derivatives $D_j f_i$ exists and are continuous on E for $1 \le i \le m$, $1 \le j \le n$. **6+5+3**

(b) If $T \in L(\mathbf{R}^n, \mathbf{R}^m)$, then prove that $\|T\| < \infty$ and T is a uniformly continuous mapping of $\mathbf{R}^n$ onto $\mathbf{R}^m$.

(c) Let $f: [a, b] \to \mathbf{R}^k$, $f = (f_1, f_2, \ldots\ldots, f_k)$, f is differentiable if and only if each f_i is differentiable.

8. State and prove the implicit function theorem. **14**

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